Therefore (2) holds (with strict inequality) when $a \neq b$. Moreover, if $a = b(> 0)$, then both sides of (2) equal a, so (2) becomes an equality. This proves that (2) holds for $a > 0, b > 0$.

On the other hand, suppose that $a > 0$, $b > 0$ and that $\sqrt{ab} = \frac{1}{2}(a + b)$. Then, squaring both sides and multiplying by 4, we obtain

$$4ab = (a + b)^2 = a^2 + 2ab + b^2,$$

whence it follows that

$$0 = a^2 - 2ab + b^2 = (a - b)^2.$$

But this equality implies that $a = b$. (Why?) Thus, equality in (2) implies that $a = b$.

Remark The general Arithmetic-Geometric Mean Inequality for the positive real numbers $a_1, a_2, \ldots, a_n$ is

(3) $$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

with equality occurring if and only if $a_1 = a_2 = \cdots = a_n$. It is possible to prove this more general statement using Mathematical Induction, but the proof is somewhat intricate. A more elegant proof that uses properties of the exponential function is indicated in Exercise 8.3.9 in Chapter 8.

(c) Bernoulli's Inequality. If $x > -1$, then

(4) $$(1 + x)^n \geq 1 + nx \qquad \text{for all} \quad n \in \mathbb{N}$$

The proof uses Mathematical Induction. The case $n = 1$ yields equality, so the assertion is valid in this case. Next, we assume the validity of the inequality (4) for $k \in \mathbb{N}$ and will deduce it for $k + 1$. Indeed, the assumptions that $(1 + x)^k \geq 1 + kx$ and that $1 + x > 0$ imply (why?) that

$$\begin{aligned} (1 + x)^{k+1} &= (1 + x)^k \cdot (1 + x) \\ &\geq (1 + kx) \cdot (1 + x) = 1 + (k + 1)x + kx^2 \\ &\geq 1 + (k + 1)x. \end{aligned}$$

Thus, inequality (4) holds for $n = k + 1$. Therefore, (4) holds for all $n \in \mathbb{N}$. □

Exercises for Section 2.1

1. If $a, b \in \mathbb{R}$, prove the following.
   (a) If $a + b = 0$, then $b = -a$,
   (b) $-(-a) = a$,
   (c) $(-1)a = -a$,
   (d) $(-1)(-1) = 1$.
2. Prove that if $a, b \in \mathbb{R}$, then
   (a) $-(a + b) = (-a) + (-b)$,
   (b) $(-a) \cdot (-b) = a \cdot b$,
   (c) $1/(-a) = -(1/a)$,
   (d) $-(a/b) = (-a)/b$ if $b \neq 0$.
3. Solve the following equations, justifying each step by referring to an appropriate property or theorem.
   (a) $2x + 5 = 8$,
   (b) $x^2 = 2x$,
   (c) $x^2 - 1 = 3$,
   (d) $(x - 1)(x + 2) = 0$.

4. If $a \in \mathbb{R}$ satisfies $a \cdot a = a$, prove that either $a = 0$ or $a = 1$.
5. If $a \neq 0$ and $b \neq 0$, show that $1/(ab) = (1/a)(1/b)$.
6. Use the argument in the proof of Theorem 2.1.4 to show that there does not exist a rational number s such that $s^2 = 6$.
7. Modify the proof of Theorem 2.1.4 to show that there does not exist a rational number t such that $t^2 = 3$.
8. (a) Show that if x, y are rational numbers, then $x + y$ and xy are rational numbers.
   (b) Prove that if x is a rational number and y is an irrational number, then $x + y$ is an irrational number. If, in addition, $x \neq 0$, then show that xy is an irrational number.
9. Let $K := \{s + t\sqrt{2} : s, t \in \mathbb{Q}\}$. Show that K satisfies the following:
   (a) If $x_1, x_2 \in K$, then $x_1 + x_2 \in K$ and $x_1 x_2 \in K$.
   (b) If $x \neq 0$ and $x \in K$, then $1/x \in K$.
   (Thus the set K is a *subfield* of $\mathbb{R}$. With the order inherited from $\mathbb{R}$, the set K is an ordered field that lies between $\mathbb{Q}$ and $\mathbb{R}$.)
10. (a) If $a < b$ and $c \leq d$, prove that $a + c < b + d$.
    (b) If $0 < a < b$ and $0 \leq c \leq d$, prove that $0 \leq ac \leq bd$.
11. (a) Show that if $a > 0$, then $1/a > 0$ and $1/(1/a) = a$.
    (b) Show that if $a < b$, then $a < \frac{1}{2}(a + b) < b$.
12. Let a, b, c, d be numbers satisfying $0 < a < b$ and $c < d < 0$. Give an example where $ac < bd$, and one where $bd < ac$.
13. If $a, b \in \mathbb{R}$, show that $a^2 + b^2 = 0$ if and only if $a = 0$ and $b = 0$.
14. If $0 \leq a < b$, show that $a^2 \leq ab < b^2$. Show by example that it does *not* follow that $a^2 < ab < b^2$.
15. If $0 < a < b$, show that (a) $a < \sqrt{ab} < b$, and (b) $1/b < 1/a$.
16. Find all real numbers x that satisfy the following inequalities.
    (a) $x^2 > 3x + 4$,
    (b) $1 < x^2 < 4$,
    (c) $1/x < x$,
    (d) $1/x < x^2$.
17. Prove the following form of Theorem 2.1.9: If $a \in \mathbb{R}$ is such that $0 \leq a \leq \varepsilon$ for every $\varepsilon > 0$, then $a = 0$.
18. Let $a, b \in \mathbb{R}$, and suppose that for every $\varepsilon > 0$ we have $a \leq b + \varepsilon$. Show that $a \leq b$.
19. Prove that $\left[\frac{1}{2}(a + b)\right]^2 \leq \frac{1}{2}(a^2 + b^2)$ for all $a, b \in \mathbb{R}$. Show that equality holds if and only if $a = b$.
20. (a) If $0 < c < 1$, show that $0 < c^2 < c < 1$.
    (b) If $1 < c$, show that $1 < c < c^2$.
21. (a) Prove there is no $n \in \mathbb{N}$ such that $0 < n < 1$. (Use the Well-Ordering Property of $\mathbb{N}$.)
    (b) Prove that no natural number can be both even and odd.
22. (a) If $c > 1$, show that $c^n \geq c$ for all $n \in \mathbb{N}$, and that $c^n > c$ for $n > 1$.
    (b) If $0 < c < 1$, show that $c^n \leq c$ for all $n \in \mathbb{N}$, and that $c^n < c$ for $n > 1$.
23. If $a > 0, b > 0$, and $n \in \mathbb{N}$, show that $a < b$ if and only if $a^n < b^n$. [*Hint:* Use Mathematical Induction.]
24. (a) If $c > 1$ and $m, n \in \mathbb{N}$, show that $c^m > c^n$ if and only if $m > n$.
    (b) If $0 < c < 1$ and $m, n \in \mathbb{N}$, show that $c^m < c^n$ if and only if $m > n$.
25. Assuming the existence of roots, show that if $c > 1$, then $c^{1/m} < c^{1/n}$ if and only if $m > n$.
26. Use Mathematical Induction to show that if $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$, then $a^{m+n} = a^m a^n$ and $(a^m) = a^{mn}$.